Critical thermodynamics of the two-dimensional systems in five-loop renormalization-group approximation

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Abstract

The RG functions of the $2D$ n-vector $\lambda\phi^4$ model are calculated in the five-loop approximation. Perturbative series for the β-function and critical exponents are resummed by the Pade-Borel and Pade-Borel-Leroy techniques, resummation procedures are optimized and an accuracy of the numerical results is estimated. In the Ising case $n = 1$ as well as in the others ($n = 0$, $n = -1$, $n = 2, 3, ...32$) an account for the five-loop term is found to shift the Wilson fixed point location only briefly, leaving it outside the segment formed by the results of the corresponding lattice calculations; even error bars of the RG and lattice estimates do not overlap in the most cases studied. This is argued to reflect the influence of the singular (non-analytical) contribution to the β-function that can not be found perturbatively. The evaluation of the critical exponents for $n = 1$, $n = 0$ and $n = -1$ in the five-loop approximation and comparison of the numbers obtained with their known exact counterparts confirm the conclusion that non-analytical contributions are visible in two dimensions. For the $2D$ Ising model, the estimate $\omega = 1.31(3)$ for the correction-to-scaling exponent is found that is close to the value $4/3$ resulting from the conformal invariance.

What follows is the radically shortened version of the paper written in Russian and published in "Fizika Tverdogo Tela" 42, No 11, 2087 (2000). It contains all the formulas, tables and complete list of references of the original paper available now via http://www.ioffe.rssi.ru/journals/ftt.

The Hamiltonian of the model describing the critical behavior of various two-dimensional systems reads:

$$H = \int d^2x \left[\frac{1}{2}(m_0^2 \varphi_\alpha^2 + (\nabla \varphi_\alpha)^2) + \frac{\lambda}{24}(\varphi_\alpha^2)^2 \right], \tag{1}$$

where φ_α is a real n-vector field, m_0^2 is proportional to $T - T_c^{(0)}$, $T_c^{(0)}$ being the mean-field transition temperature.

We calculate the β-function and the critical exponents for the model (1) within the massive theory. The Green function, the four-point vertex and the ϕ^2 insertion are normalized in a conventional way:

$$
\begin{aligned}
G_R^{-1}(0,m,g_4) &= m^2, \qquad & \frac{\partial G_R^{-1}(p,m,g_4)}{\partial p^2}\bigg|_{p^2=0} &= 1, \\
\Gamma_R(0,0,0,m,g) &= m^2 g_4, \qquad & \Gamma_R^{1,2}(0,0,m,g_4) &= 1.
\end{aligned}
\tag{2}
$$

Since the four-loop RG expansions at $n=1$ are known [1] we are in a position to find corresponding series for arbitrary n and to calculate the five-loop terms. The results of our calculations are as follows:

$$
\begin{aligned}
\frac{\beta(g)}{2} &= -g + g^2 - \frac{g^3}{(n+8)^2}\Big(10.33501055\ n + 47.67505273\Big) \\
&+ \frac{g^4}{(n+8)^3}\Big(5.000275928\ n^2 + 149.1518586\ n + 524.3766023\Big) \\
&- \frac{g^5}{(n+8)^4}\Big(0.088842906\ n^3 + 179.6975910\ n^2 + 2611.154798\ n + 7591.108694\Big) \\
&+ \frac{g^6}{(n+8)^5}\Big(-0.00407946\ n^4 + 80.3096\ n^3 + 5253.56\ n^2 + 53218.6\ n + 133972\Big).
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
\gamma^{-1} &= 1 - \frac{n+2}{n+8}\ g + \frac{g^2}{(n+8)^2}\ (n+2)\ 3.375628955 \\
&- \frac{g^3}{(n+8)^3}\Big(4.661884772\ n^2 + 34.41848329\ n + 50.18942749\Big) \\
&+ \frac{g^4}{(n+8)^4}\Big(0.318993036\ n^3 + 71.70330240\ n^2 + 429.4244948\ n + 574.5877236\Big) \\
&- \frac{g^5}{(n+8)^5}\Big(0.0938051\ n^4 + 85.4975\ n^3 + 1812.19\ n^2 + 8453.70\ n + 10341.1\Big).
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
\eta &= \frac{g^2}{(n+8)^2}\ (n+2)\ 0.9170859698 - \frac{g^3}{(n+8)^2}\ (n+2)\ 0.05460897758 \\
&+ \frac{g^4}{(n+8)^4}\Big(-0.0926844583\ n^3 + 4.05641051\ n^2 + 29.2511668\ n + 41.5352155\Big) \\
&- \frac{g^5}{(n+8)^5}\Big(0.0709196\ n^4 + 1.05240\ n^3 + 57.7615\ n^2 + 325.329\ n + 426.896\Big).
\end{aligned}
\tag{5}
$$

Instead of the renormalized coupling constant g_4, a rescaled coupling

$$
g = \frac{n+8}{24\pi} g_4,
\tag{6}
$$

is used as an argument in above RG series. This variable is more convenient since it does not go to zero under $n \to \infty$ but approaches the finite value equal to unity.

To evaluate the Wilson fixed point location g^* and numerical values of the critical exponents, the resummation procedure based on the Borel-Leroy transformation

$$f(x) = \sum_{i=0}^{\infty} c_i x^i = \int_0^{\infty} e^{-t} t^b F(xt) dt, \qquad F(y) = \sum_{i=0}^{\infty} \frac{c_i}{(i+b)!} y^i \quad , \tag{7}$$

is used. The analytical extension of the Borel transforms is performed by exploiting relevant Padé approximants [L/M]. In particular, four subsequent diagonal and near-diagonal approximants [1/1], [2/1], [2/2], and [3/2] turn out to lead to numerical estimates for g^* which rapidly converge, via damped oscillations, to the asymptotic values (see Table I). As is seen from Table II, these asymptotic values, however, differ appreciably from numerical estimates for g^* given by the lattice and Monte Carlo calculations; such estimates are usually extracted from the data obtained for the linear (χ) and non-linear (χ_4) susceptibilities related to each another via g_4:

$$\chi_4 = \frac{\partial^3 M}{\partial H^3}\bigg|_{H=0} = -\chi^2 m^{-2} g_4, \tag{8}$$

An account for higher-order (six-loop, seven-loop, etc.) terms in the RG expansion (3) will not avoid this discrepancy which is thus believed to reflect the influence of the singular (non-analytical) contribution to the β-function.

The critical exponents for the Ising model ($n = 1$) and for those with $n = 0$ and $n = -1$ are estimated by the Padé-Borel summation of the five-loop expansions (4), (5) for γ^{-1} η. Both the five-loop RG (Table I) and the lattice (Table II) estimates for g^* are used in the course of the critical exponent evaluation. To get an idea about an accuracy of the numerical results obtained the exponents are estimated using different Padé approximants, under various values of the shift parameter b, etc. In particular, the exponent η is estimates in two principally different ways: by direct summation of the series (5) and via the resummation of RG expansions for exponents

$$\eta^{(2)} = \frac{1}{\nu} + \eta - 2, \qquad \eta^{(4)} = \frac{1}{\nu} - 2, \tag{9}$$

which possess a regular structure favoring the rapid convergence of the iteration procedure. The typical error bar thus found is about 0.05.

The results obtained are collected in Table III. As is seen, for small exponent η and in some other cases the differences between the five-loop RG estimates and known exact values of the critical exponents exceed the error bar mentioned. Moreover, in the five-loop approximation the correction-to-scaling exponent ω of the 2D Ising model is found to be close to the value 4/3 predicted by the conformal theory but differs markedly from the exact value $\omega = 1$ [33]. This confirms the conclusion that non-analytical contributions are visible in two dimensions.

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TABLES

TABLE I. The Wilson fixed point coordinate for models with $n = 1$, $n = 0$ and $n = -1$ in four subsequent RG approximations and the final five-loop estimates for $g^*(n)$.

n	[1/1]	[2/1]	[2/2]	[3/2]	g^*, 5-loop
1	2.4246	1.7508	1.8453	1.8286	1.837 $\pm$ 0.03
0	2.5431	1.7587	1.8743	1.8402	1.86 $\pm$ 0.04
-1	2.6178	1.7353	1.8758	1.8278	1.85 $\pm$ 0.05

TABLE II. The Wilson fixed point coordinate g^* and critical exponent ω for $-1 \le n \le 32$ obtained in the five-loop RG approximation. The values of g^* extracted from high-temperature (HT) and strong coupling (SC) expansions, found by Monte Carlo simulations (MC), obtained by the constrained resummation of the ϵ-expansion for g^* (ϵ-exp.), and given by corresponding $1/n$-expansion ($1/n$-exp.) are also presented for comparison.

n	-1	0	1	2	3	4	8	16	32
g^*									
RG, 5-loop	1.85(5)	1.86(4)	1.837(30)	1.80(3)	1.75(2)	1.70(2) ($b = 1$)	1.52(1) ($b = 1$)	1.313(3) ([4/1], [3/1])	1.170(2) ([4/1], [3/1])
HT exp. [22,24]		1.679(3)	1.754(1)	1.81(1)	1.724(9)	1.655(16)			
MC [25,29]			1.71(12)	1.76(3)	1.73(3)				
SC [23]	1.473(8)	1.673(8)	1.746(8)	1.81(2)	1.73(4)				
ϵ-exp. [24]		1.69(7)	1.75(5)	1.79(3)	1.72(2)	1.64(2)	1.45(2)	1.28(1)	1.16(1)
1/n-exp. [24]					1.758	1.698	1.479	1.283	1.154
ω									
RG, 5-loop	1.32(4)	1.31(3)	1.31(3)	1.32(3)	1.33(2)	1.37(3)	1.50(2)	1.70(1)	1.85(2)

TABLE III. Critical exponents for $n = 1$, $n = 0$, and $n = -1$ obtained via the Padé-Borel summation of the five-loop RG expansions for γ^{-1} and η. The known exact values of these exponents are presented for comparison.

n		g^*	γ	η	ν	α	β
1	RG	1.837	1.779	0.146	0.960	0.081	0.070
		1.754 (HT)	1.739	0.131	0.931	0.139	0.061
	exact		7/4	1/4	1	0	1/8
			(1.75)	(0.25)			(0.125)
0	RG	1.86	1.449	0.128	0.774	0.452	0.049
		1.679 (HT)	1.402	0.101	0.738	0.524	0.037
	exact		43/32	5/24	3/4	1/2	5/64
			(1.34375)	(0.20833)	(0.75)	(0.5)	(0.078125)
-1	RG	1.85	1.184	0.082	0.617	0.765	0.025
		1.473 (SC)	1.155	0.049	0.592	0.816	0.014
	exact		37/32	3/20	5/8	3/4	3/64
			(1.15625)	(0.15)	(0.625)	(0.75)	(0.046875)